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Variational Approach to the Hamiltonian Structure of a New Hierarchy of Nonlinear Equations

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We determine the symplectic Hamiltonian structure associated with the nonlinear evolution equations obtained from two new isospectral problems. We follow the method of variation with respect to the field variables. An explicit example is given to demonstrate the new class of equations that are generated.

Of late several class of nonlinear equations have been deduced from different spectral problems and their Hamiltonian forms are deduced from the variational principle (Tu Gui Zhang, 1982a) or with the help of squared eigenfunction (Newell, 19??) and scattering data. Such Hamiltonian forms display the symplectic structure and the Poisson bracket generating the flow. They can also serve as the source for the future quantization of the nonlinear system through the idea of deformation of the Poisson bracket (Nierderle, 1976). Here we have discussed the nonlinear equations associated with a new spectral problem (and later a modified version of it) and subsequently the canonical structure for both of them from the variational techniques put forward by Tu Gui Zhang (1981a, b; 1983; 1982b).

The eigenvalue problem under discussion[2] reads

$$y_x = \begin{pmatrix} -i\lambda + r & p \\ q & i\lambda - r \end{pmatrix} y \tag{1}$$

where y is a two-component vector. If we associate one equation for the time variation of y along with (1) then we can generate a set of three coupled

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[2]Need for such an isospectral problem was described in a paper by Jaulet (1982).

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nonlinear evolution equations for the three fields (r, p. q). Let us set

$$y_t = vy = \sum_{j=0}^{n} v_j \lambda^{n-j} \tag{2}$$

with

$$v_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \tag{3}$$

The compatibility condition between (1) and (3) yields

$$U_t - V_x + [u, v] = 0 \tag{4}$$

which leads to (by equating various powers of λ) the following recurrence relation for the coefficients (a_j, b_j, c_j) as

$$\begin{pmatrix} a_{j+1} \\ b_{j+1} \\ c_{j+1} \end{pmatrix} = \begin{pmatrix} iD^{-1}(rD) & \frac{1}{2}iD^{-1}(qD) & \frac{i}{2}D^{-1}(pD) \\ ip & r-\frac{1}{2}D & 0 \\ iq & 0 & r+\frac{1}{2}D \end{pmatrix} \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} \tag{5}$$

and the nonlinear equations generated are written as

$$\begin{aligned} r_t &= a_{nx} + pc_n - qb_n \\ p_t &= 2ib_{n+1} \\ q_t &= -2ic_{n+1} \end{aligned} \tag{6}$$

Combining equations (5) and (6) we get that the evolution quations can be written

$$\begin{pmatrix} r_t \\ p_t \\ q_t \end{pmatrix} = J_0 L^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} \tag{7}$$

where

$$J_0 = \begin{pmatrix} \partial/\partial x & -q & p \\ 29 & 2r - \partial/\partial x & 0 \\ -2q & 0 & -2r - \partial/\partial x \end{pmatrix} \tag{8}$$

and

$$L = \begin{pmatrix} i\int r\,\partial/\partial x & \frac{i}{2}\int q\,\partial/\partial x & \frac{i}{2}\int p\,\partial/\partial x \\ ip & r - \frac{1}{2}\partial/\partial x & 0 \\ iq & 0 & r + \frac{1}{2}\partial/\partial x \end{pmatrix} \tag{9}$$

Our next problem is to show that the quantities a_j, b_j, c_j all can be written as the variational derivatives of some conserved densities to be used as Hamiltonians. For this purpose let us consider the Riccati equation obtained by setting $Z = y_2/y_1$ which is

$$Z_x = q + 2(i\lambda - r)Z - pZ^2 \tag{10}$$

Let us set $Z = \sum_{n=1}^{\alpha} Z_n \lambda^{-n}$ in (10) so that equating powers of λ, we get

$$Z_1 = -\frac{q}{2i} \tag{11}$$

along with

$$Z_2 = \frac{q_x}{4} + \frac{qr}{2}$$

$$Z_3 = -\frac{q_{xx}}{ri} - \frac{qr_x}{4i} - \frac{2rq_x}{4i} - \frac{qr^2}{2i}$$

with the following relation determining higher-order $Z_j'S_j$

$$Z_{mx} = 2iZ_{m+1} - 2rZ_m - p\sum_{l=1}^{m} Z_l Z_{m-l} \tag{12}$$

Now the Hamiltonians are defined as

$$H = -i\lambda + r + pZ \tag{13}$$

So if we consider

$$H = \sum_{n=-1}^{\alpha} H_n \lambda^{-n}$$

then

$$H_{-1} = -i$$

$$H_0 = r$$

$$H_1 = \frac{pq}{2i} \tag{14}$$

$$H_2 = -\frac{pq_x}{4} - \frac{pqr}{2}$$

and so on.

Now let us define a quantity as follows:

$$R = pZ \tag{15}$$

Then equations (10), (13), and (15) form the basis of further calculation. By considering the variational derivatives of these equations with respect H, Z, and R we obtain

$$\left.\begin{aligned} V_H(r) &= 1 \\ V_H(q) &= 2Z \\ V_H(p) &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} V_Z(r) &= 0 \\ V_Z(q) &= -D - 2i\lambda + 2r + pZ \\ V_Z(p) &= -p/Z \end{aligned}\right\}$$

along with

$$\left.\begin{aligned} V_R(p) &= 1/Z \\ V_R(r) &= -1 \\ V_R(q) &= -Z \end{aligned}\right\} \tag{16}$$

We now use the chain rule for the variational derivatives, written

$$\begin{pmatrix} \delta/\delta H \\ \delta/\delta Z \\ \delta/\delta R \end{pmatrix} = \begin{pmatrix} V_H(r) & V_H(p) & V_H(q) \\ V_Z(r) & V_Z(p) & V_Z(q) \\ V_R(r) & V_R(p) & V_R(q) \end{pmatrix} \begin{pmatrix} \delta/\delta r \\ \delta/\delta p \\ \delta/\delta q \end{pmatrix} \tag{17}$$

which yields

$$= \begin{pmatrix} 1 & 0 & 2Z \\ 0 & -p/Z & -D + H - i\lambda + r \\ -1 & 1/Z & -Z \end{pmatrix} \begin{pmatrix} \delta/\delta r \\ \delta/\delta p \\ \delta/\delta q \end{pmatrix} \tag{18}$$

Now if we denote the variational derivatives of the Hamiltonians with respect to (r, p, q), as M, N, and Q then (18) yields after some manipulation the following differential equationfor the set (M, N, Q):

$$(\text{grad } H)_n = \begin{pmatrix} M \\ N \\ Q \end{pmatrix}_n = \begin{pmatrix} 0 & 2p & -2q \\ q & 2i\lambda - 2r & 0 \\ -p & 0 & -2i\lambda + 2r \end{pmatrix} \begin{pmatrix} M \\ N \\ Q \end{pmatrix} \tag{19}$$

Now to achieve the aforesaid identification we expand the (grad H) in power series of λ, that is,

$$M = \sum M_j \lambda^{n-j}, \qquad N = \sum N_j \lambda^{n-j}$$
$$Q = \sum Q_j \lambda^{n-j}$$

in both sides of (19), and equate powers of λ, to obtain

$$\begin{pmatrix} M_{j+1} \\ Q_{j+1} \\ N_{j+1} \end{pmatrix} = \begin{pmatrix} i\int r\,\partial/\partial x & i\int q\,\partial/\partial x & i\int p\,\partial/\partial x \\ -\frac{ip}{2} & -\frac{1}{2}\partial/\partial x + r & 0 \\ -\frac{iq}{2} & 0 & \frac{1}{2}\partial/\partial x + r \end{pmatrix} \begin{pmatrix} M_j \\ Q_j \\ N_j \end{pmatrix} \tag{20}$$

The coefficients occurring in the expansion of the gradients of the Hamiltonian follow a similar law of formation and it is easily observed that from equations (20) and (9) that we can have the identification

$$\begin{pmatrix} a_j \\ b_j/2 \\ c_j/2 \end{pmatrix} = \alpha \begin{pmatrix} \delta H_j/\delta r \\ \delta H_j/\delta q \\ \delta H_j/\delta p \end{pmatrix} \tag{21}$$

Therefore we can write the jth flow corresponding to equation (1) as

$$\begin{pmatrix} r_t \\ p_t \\ q_t \end{pmatrix} = \alpha \begin{pmatrix} \partial/\partial x & -q & p \\ 2p & 2r - \partial/\partial r & 0 \\ 2q & 0 & -2r - \partial/\partial r \end{pmatrix} \begin{pmatrix} \delta H_j/\delta r \\ 2\delta H_j/\delta q \\ 2\delta H_j/\delta p \end{pmatrix} \tag{22}$$

which is the final form of the equation of motion written with the help of a Hamiltonian structure. At this point it is not out of place to state that we can generate another class of equations by a simple change in the eigenvalue problem, which now reads

$$y_r = \begin{pmatrix} -i\lambda + r & \lambda p \\ q & i\lambda - r \end{pmatrix} y \tag{23}$$

Proceeding as before we set

$$y_t = Vy \qquad \text{with } V = \sum V_j \lambda^{n-j} \tag{24}$$

and

$$V_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$$

Applying the same consistency condition (4) we get the following form of recursion relation and the specific type of nonlinear equation hierarchy solvable with the help of the isospectral problem (23).

The recurrence is written

$$\begin{pmatrix} a_j \\ b_j \\ l_j \end{pmatrix} = \begin{bmatrix} -\frac{1}{2}D^{-1}q\sigma & (1/2i)\,D^{-1}\sigma & D^{-1}\eta/4 \\ pD^{-1}q\sigma & 1/2i(D+2r) - (p/i)\,D^{-1}\mu & -(p/2)\,D^{-1}\eta \\ -q/i & 0 & 1/2i(-D+2r) \end{bmatrix} \begin{pmatrix} a_{j-1} \\ b_{j-1} \\ c_{j-1} \end{pmatrix} \tag{25}$$

where

$$\begin{aligned}\sigma &= -pD+2rp\\ \mu &= qD+2rq\\ \eta &= pD^2-4rpD+4r^2p\end{aligned} \tag{26}$$

The operator occurring in the square bracket in equation (25) will be referred to as L' in the following. The equations that are generated are given as

$$\begin{pmatrix} r_t \\ p_t \\ q_t \end{pmatrix} = \begin{pmatrix} ipq & 0 & (ip/2)(-D+2r) \\ 2p & 2i & 0 \\ 2q & 0 & D-2r \end{pmatrix} L_n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} \tag{27}$$

It is now very straightforward to proceed as in equation (12) to set up the corresponding Riccati equation and use an expansion of the form

$$H = i\lambda - r + p\sum_{1}^{\alpha} Z_n\lambda^{-n+1} \tag{28}$$

to generate a series of Hamiltonians, which are given as

$$\begin{aligned} H_{-1} &= i\\ H_0 &= r\\ H_1 &= \frac{pq_r}{4} - \frac{pqr}{2} + \frac{p^2q^2}{8i}\\ H_2 &= -\frac{pq_{rr}}{8i} + \frac{prq_r}{2i} + \frac{pqr_r}{2} + \frac{pp_rq^2}{16} + \frac{3p^2qq_r}{16}\\ &\quad + \frac{p^3q^3}{16i} - \frac{pqr^2}{2i} - \frac{3rp^2q^2}{8}\end{aligned}$$

and many more through the relations

$$H_m = pZ_{m+1}, \qquad m \geq 1$$

and

$$Z_{mr} = -2iZ_{m+1} + 2rZ_m - p\sum_{l=1}^{m} Z_lZ_{m-l+1} \tag{29}$$

Proceeding as in the previous calculation we can show that the gradients of the Hamiltonians determined through (28) satisfy an equation similar to (20) is which upon expansion in λ yields a recursion relation similar to those of (a_j, b_j, c_j), that is, equation (25), thus proving the gradient like structure for the heirarchy.

Let us now consider a particular case of equation (23). Let us consider a second-order flow with the following specifications:

$$r = -\frac{i}{2}u, \qquad p = iv, \qquad q = -i \tag{30}$$

and

$$a_2 = \frac{i}{2}u^2 - \frac{1}{2}u$$

$$b_2 = iuv + v_1 \tag{31}$$

$$c_2 = -iu$$

then the nonlinear equations generated are a pair of coupled nonlinear Schrödinger equations:

$$\begin{aligned} iu_t &= 2iuu_x - u_{xx} + 2iv_x \\ iv_t &= 2i(uv)_x + v_{xx} \end{aligned} \tag{32}$$

It is interesting to note that such a pair was initially referred to by Novikov (19??) et al. for their periodic structure without any reference to the form of their *IST* equations. So our isospectral problems really generate an extension of the usual AKNS system to more complex nature and to more complicated type of coupling.

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